Lecture 25. Linear dynamical systems

Def A <u>linear dynamical system</u> is an equation $\vec{x}_{n+1} = A\vec{x}_n$ for a square matrix A and a sequence of vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots$.

Note (1) The vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots$ are called the <u>state vectors</u>, which may represent a collection of quantities that change over time.

e.g. stock prices, city populations, $\cdots$

(2) If the initial state $\vec{x}_0$ is known, the system has a unique solution given by $\vec{x}_n = A^n \vec{x}_0$.

$(\vec{x}_1 = A\vec{x}_0,\ \vec{x}_2 = A\vec{x}_1 = AA\vec{x}_0 = A^2\vec{x}_0,\ \vec{x}_3 = A\vec{x}_2 = AA^2\vec{x}_0 = A^3\vec{x}_0, \cdots)$

Prop If a square matrix A has a diagonalization $A = PDP^{-1}$, its power A^n has a diagonalization $A^n = PD^nP^{-1}$

e.g. $A^2 = (PDP^{-1})^2 = PD\cancel{P^{-1}P}DP^{-1} = PD^2P^{-1}$

$A^3 = (PDP^{-1})^3 = PD\cancel{P^{-1}P}D\cancel{P^{-1}P}DP^{-1} = PD^3P^{-1}$

Prop Given a diagonal matrix D, its power D^n is given by raising each diagonal entry of D to the n^{th} power.

e.g. $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow D^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},\ D^3 = \begin{bmatrix} 1^3 & 0 \\ 0 & 2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$

Note If a square matrix A has a diagonalization, it is easy to find the solution of the dynamical system $\vec{x}_{n+1} = A\vec{x}_n$.

Ex Find the solution of the dynamical system $\vec{x}_{n+1} = A\vec{x}_n$ with

$$A = \begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix} \text{ and } \vec{x}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Sol $p_A(\lambda) = \lambda^2 - (4-1)\lambda + (4\cdot(-1) - (-6)\cdot 1) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

$\Rightarrow$ A has eigenvalues $\lambda = 1, 2$

$\Rightarrow$ A is diagonalizable (2×2 matrix with 2 distinct eigenvalues)

$$A - I = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \Rightarrow \text{RREF}(A-I) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$(A-I)\vec{x} = \vec{0} \Rightarrow x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2 \underset{x_2 = t}{\Rightarrow} \vec{x} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Nul$(A-I)$ has a basis given by $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A - 2I = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \Rightarrow \text{RREF}(A-2I) = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$(A-2I)\vec{x} = \vec{0} \Rightarrow x_1 - 3x_2 = 0 \Rightarrow x_1 = 3x_2 \underset{x_2 = t}{\Rightarrow} \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Nul$(A-2I)$ has a basis given by $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Hence A has a diagonalization $A = PDP^{-1}$ with

$$P = \begin{array}{c} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ \vec{v} \quad \vec{w} \end{array} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

eigenvalues for $\vec{v}, \vec{w}$

$$\Rightarrow A^n = PD^nP^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\left(P^{-1} = \frac{1}{2\cdot 1 - 3\cdot 1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \right)$$

$$\Rightarrow \vec{x}_n = A^n \vec{x}_0 = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}}_{A^n}\underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\vec{x}_0}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2^n \end{bmatrix} = \boxed{\begin{bmatrix} 2+3\cdot 2^n \\ 1+2^n \end{bmatrix}}$$

<u>Note</u> Alternatively, we may compute $\vec{x}_n = A^n \vec{x}_0$ by writing $\vec{x}_0$ as a linear combination of $\vec{v}$ and $\vec{w}$.

We want $\vec{x}_0 = y_1\vec{v} + y_2\vec{w} \Rightarrow \vec{x}_0 = P\vec{y}$

$$\begin{array}{c} \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array}\right] \\ \quad P \qquad \vec{x}_0 \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow y_1 = 1,\ y_2 = 1 \Rightarrow \vec{x}_0 = \vec{v} + \vec{w}$$

$$\Rightarrow \vec{x}_n = A^n\vec{x}_0 = A^n(\vec{v}+\vec{w}) = A^n\vec{v} + A^n\vec{w} = \vec{v} + 2^n\vec{w}$$

$$\left\{ \begin{array}{l} A\vec{v} = \vec{v} \Rightarrow A^n\vec{v} = 1^n\vec{v} = \vec{v} \\ A\vec{w} = 2\vec{w} \Rightarrow A^n\vec{w} = 2^n\vec{w} \end{array} \right.$$

$$\Rightarrow \vec{x}_n = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2^n\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+3\cdot 2^n \\ 1+2^n \end{bmatrix}$$

$\underline{\text{Ex}}$ If possible, evaluate $\lim_{n\to\infty} \vec{x}_n$ for the dynamical system $\vec{x}_{n+1} = A\vec{x}_n$ with

$$A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \text{ and } \vec{x}_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}.$$

$\underline{\text{Sol}}$ $p_A(\lambda) = \lambda^2 - (0.8+0.4)\lambda + (0.8\cdot 0.4 - 0.6\cdot 0.2) = \lambda^2 - 1.2\lambda + 0.2 = (\lambda-1)(\lambda-0.2)$

$\Longrightarrow$ A has eigenvalues $\lambda = 1, 0.2$

$\Longrightarrow$ A is diagonalizable (2×2 matrix with 2 distinct eigenvalues)

$$A - I = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} \Longrightarrow \text{RREF}(A-I) = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$(A-I)\vec{x} = \vec{0} \Longrightarrow x_1 - 3x_2 = 0 \Longrightarrow x_1 = 3x_2 \underset{x_2=t}{\Longrightarrow} \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ Nul$(A-I)$ has a basis given by $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$A - 0.2I = \begin{bmatrix} 0.6 & 0.6 \\ 0.2 & 0.2 \end{bmatrix} \Longrightarrow \text{RREF}(A-0.2I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A-0.2I)\vec{x} = \vec{0} \Longrightarrow x_1 + x_2 = 0 \Longrightarrow x_1 = -x_2 \underset{x_2=t}{\Longrightarrow} \vec{x} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ Nul$(A-0.2I)$ has a basis given by $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence A has a diagonalization $A = PDP^{-1}$ with

$$P = \begin{array}{c} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \\ \begin{array}{cc} \vec{v} & \vec{w} \end{array} \end{array} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

eigenvalues for $\vec{v}, \vec{w}$

$$\Rightarrow A^n = PD^nP^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.2^n \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\left(P^{-1} = \frac{1}{3 \cdot 1 - 1 \cdot (-1)} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \right)$$

$$\Rightarrow \vec{x}_n = A^n \vec{x}_0 = \underbrace{\frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.2^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}}_{A^n} \underbrace{\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}}_{\vec{x}_0}$$

$$\Rightarrow \lim_{n \to \infty} \vec{x}_n = \lim_{n \to \infty} \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.2^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \lim_{n \to \infty} \begin{bmatrix} 1^n & 0 \\ 0 & 0.2^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\frac{1}{4} \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

Note This dynamical system is called a Markov chain as all columns of A and all state vectors are probability vectors (i.e., consisting of nonnegative entries that add up to 1)

For most Markov chains, the state vectors converge to a probability vector in the 1-eigenspace, as in this example.